CURVES ON CAYLEY'S DIANODAL SURFACE

BY

FRANZ EDWARD HOHN

B.S., McKendree College, 1936

M.S., University of Illinois, 1937

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1940

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CURVES ON CAYLEY'S DIANODAL SURFACE

I. INTRODUCTION

The dianodal surface Δ was first defined by Cayley in his "Memoir on Quartic Surfaces" (Proc. London Math. Soc., 3(1869) 16--69), as the locus of the eighth nodes of azygetic quartic surfaces with nodes at 7 given points. His work was extended by Rohn in Die Flächen vierter Ordnung hinsichtlich ihrer Knotenpunkte··· (Leipzig, Hirzel, 1886). Schottky defined the surface parametrically in terms of ϑ-functions of genus 3. (Journal für Math., 105(1889) 269--297.) Coble summarizes and extends these results in his Algebraic Geometry and Theta Functions (N.Y., Amer. Math. Soc., 1929). We give briefly the fundamental facts here.

The following definitions are made:

$$(1)\qquad P_{ijk} = \vartheta_{ijk8}\,\vartheta_{i8}\,\vartheta_{j8}\,\vartheta_{k8} \qquad P_{ijkl} = \pi^2\,\vartheta_{mnr8}\,/\,\vartheta_{m8}\,\vartheta_{n8}\,\vartheta_{r8}$$
$$P_{i,j} = \pi\,\vartheta_{ij}\,\vartheta_{i8}\,/\,\vartheta_{j8} \qquad P_i = \pi^2\,\vartheta_{i8}^2$$
$$(\pi = \vartheta_{18}\,\vartheta_{28}\cdots\vartheta_{78};\quad i,j,k,l,m,n,r = 1,2,\cdots,7)$$

From the 4-term identities among the 6 ϑ-products obtained from the Steiner complex associated with a given half-period, new 4-term identities are obtained by the addition of a half-period to the argument. In this way we find that there are 4 independent functions P_{ijk} and that the rest of the 35 such functions are all linear combinations of these 4. We define the surface by taking 4 independent linear combinations of 4 independent P_{ijk}'s as coordinates of a point y in S_3, for example,

$$(2)\qquad y_i = a_iP_{123} + b_iP_{124} + c_iP_{134} + d_iP_{234} \quad (i = 1,2,3,4;\ |a\ b\ c\ d| \neq 0)$$

and requiring that $\vartheta(u) = 0$. The restriction $\vartheta(u) = 0$ reduces many of the above 4-term identities to 3-term identities. From these identities, we find that: P_{ijk} represents a plane on three points, q_i, q_j, q_k of a Q_7^3 composed of the 7 points $q_i,q_j,q_k,q_l,q_m,q_n,q_r = q_1,q_2,\cdots,q_7$ of a Q_7^3 in $S_3(y)$. $P_{i,j}$ represents a cone with vertex q_i, and on all other points of Q_7^3 except q_j. P_{ijkl} represents a cubic surface with nodes at q_i,q_j,q_k,q_l and with simple points at q_m,q_n,q_r. P_i represents a quartic surface with a triple point at q_i and with double points everywhere else.

From (1) follow the identities in u, which are equations in y of the surface (2):

$$(3)\qquad P_{i,j}P_j - P_{j,i}P_i = 0 \qquad P_{i,j}P_{jlk}P_{jmnr} - P_{j,i}P_{ilk}P_{imnr} = 0$$
$$P_{i,j}P_{k,l}P_{m,n} - P_{j,i}P_{l,k}P_{n,m} = 0 \qquad P_{ijk}P_{lmn}P_r - P_{lmnr}P_{ijk} = 0$$

From these identities we obtain the following properties of the surface: (a) the surface has an elliptic triple point E_{18} at each point q_1 of Q_7^3; (b) it contains the 35 elliptic plane cubics E_{ijk8} cut out by the planes $P_{ijk} = 0$ on the cubic surfaces $P_{lmnr} = 0$; (c) it contains the 21 elliptic quartics $E_{ij} = 0$, which are on the cones $P_{i,j} = 0$ and $P_{j,i} = 0$; (d) it contains the 21 lines $F_{ij} = \overline{q_i q_j}$ of Q_7^3; and (e) it contains the 7 cubic curves F_{18} on 6 of the 7 points of Q_7^3--F_{18} is not on q_1. Further properties are given by Coble in the above-mentioned work.

Two curves C_1, C_2 on a surface F are said to be algebraically equivalent, $C_1 \equiv C_2$, if there exists an algebraic system of curves on F of which C_1 and C_2 are <u>complete</u> curves. The ρ curves $C_1, C_2, \cdots, C_\rho$ of F are said to be algebraically dependent if there exist positive integers λ , μ such that $\lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_r C_r \equiv \mu_{r+1} C_{r+1} + \cdots + \mu_\rho C_\rho$. The necessary and sufficient condition that these curves be algebraically dependent is that the <u>discriminant matrix</u> $\left\| \begin{matrix} (C_i, C_j) \\ n_i \end{matrix} \right\|$ have rank $< \rho$. Here (C_i, C_j) is the number of intersections of C_i and C_j, (C_i, C_i) the degree of C_i, and n_i the order of C_i. If $C_1, C_2, \cdots, C_\rho$ are independent and if further for every curve C on F, the $\rho + 1$ curves $C, C_1, C_2, \cdots, C_\rho$ are algebraically dependent, then the ρ curves $C_1, C_2, \cdots, C_\rho$ are said to constitute <u>a basis for the curves on F</u>. This theory of dependent curves is due to Severi. References will be found in Zariski's book, <u>Algebraic Surfaces</u>. (Berlin, Julius Springer, 1935.) In this thesis a set of curves has been found which apparently constitutes a basis for the dianodal surface Δ , but final proof that they do constitute a basis is lacking.

II. THE CURVES $F_{18}, F_{ij}, E_{18}, E_{ij}, E_{ijk8}, N^6, D^{18}$, ON Δ

From the birational point of view, the triple points of Δ are to be considered as curves of order zero, genus 1, and degree -3. Since the adjoint quartics of Δ transform the 28 curves F_{ij} and F_{18} into double points of the transformed surface, F_{ij} and F_{18} are to be regarded as having degree -2. Thus we have

$$
\begin{array}{lll}
(E_{18}, E_{18}) = -3 & (F_{ij}, F_{ik}) = 0 & (F_{18}, F_{j8}) = 0 \\
(E_{18}, E_{j8}) = 0 & (F_{ij}, F_{kl}) = 0 & (F_{18}, F_{ij}) = 0 \\
(F_{ij}, F_{ij}) = -2 & (F_{18}, F_{18}) = -2 & (F_{18}, F_{jk}) = 0
\end{array} \tag{4}
$$

Letting π be a generic plane section of Δ , we have from the section by P_{ijk}:

$$\pi \equiv F_{ij} + F_{jk} + F_{ki} + E_{i8} + E_{j8} + E_{k8} + E_{ijk8}.$$

Now, two complete curves of a given linear system intersect the same curve in the same number of points. Thus, for example:

$$(5)\quad (\pi, F_{ij}) = (F_{ij}, F_{ij}) + (F_{jk}, F_{ij}) + (F_{ki}, F_{ij}) + (E_{i8}, F_{ij}) + (E_{j8}, F_{ij}) + (E_{k8}, F_{ij}) + (E_{ijk8}, F_{ij}).$$

It is obvious that

$$(6)\quad \begin{array}{lll} (\pi, F_{ij}) = 1 & (E_{i8}, F_{ij}) = 1 & (E_{k8}, F_{ij}) = 0 \\ (\pi, E_{ijk8}) = 3 & (E_{ijk8}, E_{i8}) = 1 & (E_{ijk8}, E_{\ell 8}) = 0 \end{array}$$

Thus (5) becomes $1 = -2 + 0 + 0 + 1 + 1 + 0 + (E_{ijk8}, F_{ij})$ so that $(E_{ijk8}, F_{ij}) = 1$.

We obtain the formulae

$$(7)\quad \begin{array}{lll} (E_{ijk8}, F_{ij}) = 1 & (E_{ijk8}, F_{il}) = 0 & (E_{ijk8}, E_{ilm8}) = 1 \\ (E_{ijk8}, E_{ijk8}) = -3 & (E_{ijk8}, F_{lm}) = 1 & (E_{ijk8}, E_{lmn8}) = 0 \\ & (E_{ijk8}, E_{ij\ell 8}) = 0 & \end{array}$$

The above intersection formulae contain implicitly a geometric description of the configuration of the sections of Δ by the 35 planes $P_{ijk} = 0$. (It should not be forgotten that birationally speaking, 2 curves do not intersect at a triple point of Δ unless they have a direction in common there.)

From the sections of Δ by $P_{i,j}$; P_{ijkl} and P_l, we have the equivalences

$$(8)\quad 2\pi \equiv F_{ik} + \cdots + F_{ir} + 2E_{i8} + E_{k8} + \cdots + E_{r8} + E_{ij} + F_{j8}$$

$$(9)\quad 3\pi \equiv F_{ij} + F_{ik} + \cdots + F_{kl} + 2(E_{i8} + \cdots + E_{\ell 8}) + E_{m8} + E_{n8} + E_{r8} + E_{mnr8} + F_{m8} + F_{n8} + F_{r8}$$

$$(10)\quad 4\pi \equiv F_{ij} + \cdots + F_{ir} + 4E_{i8} + 2(E_{j8} + \cdots + E_{r8}) + F_{j8} + \cdots + F_{r8}$$

E_{i8} is counted 4 times in the last equivalence above since P_i and Δ have the same tangent cone at q_i. By the same process as before, we obtain the intersection formulae:

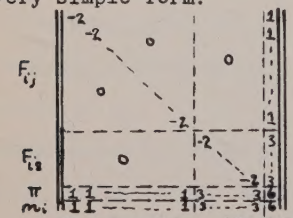
$$(11)\quad \begin{array}{lll} (E_{ij}, F_{ij}) = 0 & (E_{ij}, F_{k8}) = 0 & (E_{ijk8}, F_{i8}) = 1 \\ (E_{ij}, F_{ik}) = 1 & (E_{ij}, E_{ij}) = -3 & (E_{ijk8}, F_{\ell 8}) = 0 \\ (E_{ij}, F_{kl}) = 0 & (E_{ij}, E_{ik}) = 0 & (E_{ijk8}, E_{ij}) = 1 \\ (E_{ij}, F_{i8}) = 1 & (E_{ij}, E_{lm}) = 1 & (E_{ijk8}, E_{il}) = 0 \\ & & (E_{ijk8}, E_{lm}) = 1 \end{array}$$

The formula $(E_{ij}, F_{ik}) = 1$ corresponds to the fact that the elliptic quartics on a quadric cone have the generators as bisecants. One of the intersections in this case falls on q_k and is thus not counted. The preceding formulae contain a complete table of intersections of the $28 + 63$ rational and elliptic curves defined by Q_7^3. It may be shown that none of these

intersections occur as tangencies at the triple points, and that all of these intersections are distinct, in the generic case.

At this point it is not difficult to prove that the Cayley dianodal surface associated with a Q_7^3 is the same as the Schottky surface associated with the Q_7^3. The preceding intersection formulae imply that a sextic surface with triple points at Q_7^3, and containing the rational curves F_{ij}, F_{18}, necessarily also contains the curves E_{ijk8}, E_{ij}. Then the Riemann-Roch theorem for curves may be applied to yield the fact that there is a unique sextic containing these 28 rational curves.

The preceding formulae show that not all the $28+63+1$ curves $F_{18}, F_{ij}, E_{18}, E_{ij}, E_{ijk8}, \pi$ are independent. In fact, from the 7 equivalences (10), we can obtain E_{18} in terms of π, F_{18}, and F_{ij}. Then from the other equivalences it follows that the remaining 63 curves are dependent on the $28+1$ curves F_{18}, F_{ij}, π. The discriminant matrix of these 29 curves takes the very simple form:



The determinant of the first 29 rows has the value 3.2^{32} so that the curves F_{ij}, F_{18}, π are independent. (The curves F_{ij}, E_{is}, π, likewise may be shown to be independent.)

The Jacobian (Weddle) quartic J_i which is the locus of the vertices of the quadric cones on the 6 points of Q_7^3 obtained by omitting q_i, has these 6 points as double points, and intersects the surface in the 15 lines and the space cubic on these 6 points, and the space sextic of genus 3, N^6, which is the locus of the vertices of cones on all 7 points of Q_7^3. We have therefore the equivalence,

(12) $4\pi \equiv F_{jk} + \cdots + F_{nr} + 2(E_{j8} + \cdots + E_{r8}) + F_{18} + N^6$.

Thus the sextic N^6 is dependent on the 29 curves F_{ij}, F_{18}, π. Summing the 7 symmetric forms of (12), we obtain $28\pi \equiv 5\sum_{21} F_{ij} + 12\sum_{7} E_{18} + \sum_{7} F_{18} + 7N^6$. Summing the equivalences (10), we have $28\pi \equiv 2\sum_{21} F_{ij} + 16\sum_{7} E_{18} + 6\sum_{7} F_{18}$. Eliminating $\sum_{7} E_{18}$, we have finally

(13) $$2N^6 \equiv 2\pi - \sum_{21} F_{ij} + \sum_{7} F_{18}.$$

From (12 we obtain by the usual procedure

(14) $(N^6, E_{18}) = 0 \quad (N^6, F_{ij}) = 2 \quad (N^6, F_{18}) = 2 \quad (N^6, N^6) = -8$

$(N^6, E_{ij}) = 0 \quad (N^6, E_{ijk8}) = 0$

Since $(N^6, N^6) = -8$, we conclude that N^6 has the birational character of an 8-fold point of genus 3.

The dianodal Δ is the locus for eighth nodes of azygetic quartic surfaces with the 7 nodes Q_7^3. Let q_8 be such an eighth node. We form the dianodal surface Δ' of the 7 points $q_1, q_2, \cdots, q_6, q_8$. The intersection of Δ and Δ' is the set of 15 lines on $q_1, q_2, \cdots, q_6$, the space cubic F_{78}, and a residual curve D^{18}, which is the locus for ninth nodes of a quartic surface with the 8 nodes Q_7^3 and q_8. (Cayley.) The curve D^{18} is called a dianodal curve of Δ and has 8 triple points, one at each of $q_1, q_2, \cdots, q_7, q_8$.

The dianodal curves on D^{18} form an ∞^2 system, there being one for each point of Δ. Since the curves of the system have a variable triple point, it follows that the system of dianodal curves on Δ is not linear. We have the equivalence:

(15) $6\pi \equiv F_{12} + \cdots + F_{56} + F_{78} + 3(E_{18} + \cdots + E_{68}) + E_{78} + D^{18}$,

from which it follows that the dianodal curves of Δ are dependent on the 29 curves F_{ij}, F_{i8}, π.

We find

(16) $(D^{18}, F_{ij}) = 2$; $(D^{18}, F_{18}) = 2$; $(D^{18}, N^6) = 4$; $(D^{18}, D^{18}) = 19$

and if D'^{18} is the dianodal curve associated with q_9 on D^{18},

(17) $$(D^{18}, D'^{18}) = 19$$

From (17), since of these 19 intersections 6 lie on triple points of D^{18} and D'^{18}, it follows that the locus of tenth nodes for quartic surfaces with the 9 nodes $q_1, q_2, \cdots, q_7, q_8, q_9$ is a set of 13 points, a well-known result due to Rohn. Summing the 7 relations exemplified by (15) we have

$$42\pi \equiv 5\sum_{21} F_{ij} + \sum_{7} F_{18} + 19\sum_{7} E_{18} + 7D^{18},$$

and using again the relation $28\pi \equiv 2\sum_{21} F_{ij} + 16\sum_{7} E_{18} + 6\sum_{7} F_{18}$ to eliminate $\sum_{7} E_{18}$, we obtain

(18) $8D^{18} \equiv 10\pi - 3\sum_{21} F_{ij} + 7\sum_{7} F_{18}$.

Formulae (13) and (18) illustrate the symmetry of the curves N^6 and D^{18} with respect to the 28 rational curves of Q_7^3.

If $|C_1|$ and $|C_2|$ are linear systems with virtual genera p_1 and p_2, and if C_1 and C_2 have i virtual intersections, the virtual genus p of $|C_1 + C_2|$ is given by

(19) $$p = p_1 + p_2 + i - 1$$

Heretofore, all virtual characteristics have been the same as the effective ones. A generic plane section π of Δ has genus 10, so that the genus p of $|n\pi|$ must be given by

(20) $\qquad p = 10n + 6\cdot C(n,2) - n + 1.$

Then applying (19) repeatedly to the right member of (15), and (20) to the left, we find that if p is the virtual genus of D^{18},

$$145 = 19 + p + 143 - 35, \text{ so that } p = 18.$$

The effective genus is reduced to 15 by the (accidental) triple point q_8.

III. CURVES DEFINED BY ϑ-FUNCTIONS

The product ϑ-functions of order 2 and characteristic 0 are the 64 ϑ-squares, all even functions. The product ϑ-functions of order 2 and characteristic not 0 are as follows:

CHARACTERISTIC ij		CHARACTERISTIC ijkl	
6 of type $\vartheta_{ik}\vartheta_{jk}$	(E)	3 of type $\vartheta_{mn}\vartheta_{rs}$	(E)
(21) 15 of type $\vartheta_{mn}\vartheta_{mnij}$	(O)	3 of type $\vartheta_{ij}\vartheta_{kl}$	(E)
10 of type $\vartheta_{iklm}\vartheta_{jklm}$	(E)	9 of type $\vartheta_{mnij}\vartheta_{mnkl}$	(E)
1 of type $\vartheta\vartheta_{ij}$	(O)	1 of type $\vartheta\vartheta_{ijkl}$	(E)
		16 of type $\vartheta_{im}\vartheta_{inrs}$	(O)

When we impose the restriction $\vartheta(u) = 0$, we find that there are 7 linearly independent ϑ-squares; 4 linearly independent E-functions and 3 linearly independent O-functions of characteristic ij; and 3 independent E-functions and 4 independent O-functions of characteristic ijk8.

Multiplying each of the 64 ϑ-squares by the function π^2, we find

$$P_i = \pi^2\vartheta_{i8}^2; \qquad P_{i,j}P_{j,i} = \pi^2\vartheta_{ij}^2; \qquad P_{ijk}P_{lmnr} = \pi^2\vartheta_{ijk8}^2$$

We have then

Theorem (a): The curves determined on Δ by the ϑ-functions of order 2 and characteristic 0 are cut out by the ∞^6 adjoint quartics of Δ. The curves are of order 24 and have 6-fold points at each point of Q_7^3. From the equivalence $4\pi \equiv C^{24} + 2(\sum_7 E_{18})$ we find that these curves do not intersect the 28 rational curves F_{ij}, F_{i8}. By the method applied to the curve D^{18}, we find that these curves have genus 10.

The E-functions of characteristic 18 are the 6 of type $\vartheta_{ij}\vartheta_{j8}$ and the 10 of type $\vartheta_{ijkl}\vartheta_{jk18}$. The first 6 functions are obtained from the Steiner complex of P_{18} and hence only 3 of them are linearly independent. Adding P_{jk} to the argument in these 6 functions, we find that any two functions $\vartheta_{ijkl}\vartheta_{jk18}$ and $\vartheta_{ijkm}\vartheta_{jkm8}$ are linearly related to the 2 functions $\vartheta_{ij}\vartheta_{j8}$ and $\vartheta_{ik}\vartheta_{i8}$. It then easily follows that the 4 functions

(22) $\qquad \vartheta_{ij}\vartheta_{j8}; \qquad \vartheta_{ik}\vartheta_{k8}; \qquad \vartheta_{il}\vartheta_{l8}; \qquad \vartheta_{imnr}\vartheta_{mnr8}$

constitute a basis for these E-functions. From the definitions (1) it

follows by substitution that (22) may be expressed as the tetrad, $P_{j,i}$; $P_{k,i}$; $P_{l,i}$; $P_{jkl}P_{mnr}$, where an extraneous common factor π/ϑ_{i8} has been dropped. Thus we have

Theorem (b): The curves determined on Δ by the E-functions of order 2 and characteristic i8 are cut out by the web of quadrics

(23) $$aP_{j,i} + bP_{k,i} + cP_{l,i} + dP_{jkl}\cdot P_{mnr} = 0$$

which has a base-point at every triple point of Δ except q_1. The curves defined have order 12, triple points at the points of Q_7^3, and genus 7, as follows by the previously developed method from the equivalence

$$2\pi \equiv C^{12} + E_{j8} + E_{k8} + E_{l8} + E_{m8} + E_{n8} + E_{r8}.$$

From this we also find that the curves C^{12} meet the 12 rational curves on the seventh triple point of Δ each in one point, the others not at all.

This web of curves contains a net of curves C^{12} of genus 4 with triple points at all of the points of Q_7^3, these being cut out by the quadrics on the 7 points of Q_7^3. The quadric cones of this net cut out curves C^{12} of virtual genus 4, effective genus 3. These cones have their vertices on N^6 so that the curves C^{12} of genus 3 constitute an irrational pencil. Thus, by a theorem of Castelnuovo, Δ is irregular. The dianodal curves D^{18} associated with the points of a dianodal curve also constitute an irrational pencil, giving further proof that Δ is irregular. Unfortunately, the result is only qualitative.

From (22), by addition of the half-period P_{ij} to the argument, and from $\vartheta(u) = 0$, we find that a basis for the O-functions of characteristic i8 is given by

(24) $\vartheta_{jk}\vartheta_{ijk8}$; $\vartheta_{kl}\vartheta_{ikl8}$; $\vartheta_{lj}\vartheta_{ilj8}$ which becomes the triad, $P_{j,k}P_{l,j}P_{ijk}$; $P_{j,l}P_{l,k}P_{ikl}$; $P_{ijl}P_{j,l}P_{l,j}$, by means of (1) and (3).

Theorem (c): The curves determined on Δ by the O-functions of order 2 and characteristic i8 are cut out by the net of quintics

$$aP_{j,k}P_{l,j}P_{ijk} + bP_{j,l}P_{l,k}P_{ikl} + cP_{j,l}P_{l,j}P_{ijl} = 0$$

with the multiplicities $(i^3j^3k^2l^3m^2n^2r^2)$ and the common curves $F_{ij}, F_{jk}, F_{jl}, F_{jm}, F_{jn}, F_{jr}, F_{il}, F_{kl}, F_{lm}, F_{ln}, F_{lr}, E_{jl}$. The curves of the net have order 15 and multiplicities $(i^6j^3k^3l^3m^3n^3r^3)$. From the equivalence,

$$5\pi \equiv C^{15} + F_{ij} + F_{jk} + \cdots + F_{lr} + E_{jl} + 3E_{i8} + 3E_{j8} + 2E_{k8} + \cdots + 2E_{r8},$$

follows that these curves C^{15} meet the 12 rational curves on their 6-fold point not at all, all others each once. They have genus 6.

By essentially the same procedure used in these last 2 cases, we find that bases for the E and O functions of order 2 and characteristic ij are,

(25) $\vartheta_{ij}\vartheta_{kj}$; $\vartheta_{il}\vartheta_{lj}$; $\vartheta_{im}\vartheta_{mj}$; $\vartheta_{inr8}\vartheta_{jnr8}$

(26) $\vartheta_{kl}\vartheta_{mnr8}$; $\vartheta_{lm}\vartheta_{knr8}$; $\vartheta_{mk}\vartheta_{lnr8}$

By appropriate use of (1) and (3) the systems (25) and (26) become,

$P_{i,k}P_{k,j}$; $P_{i,l}P_{l,j}$; $P_{i,m}P_{m,j}$; $P_{inr}P_{iklm}$

$P_{l,k}P_{k,lmnr}$; $P_{m,l}P_{l,knr}$; $P_{m,k}P_{k,lnr}$

Theorem (d): The curves determined on Δ by the E-functions of order 2 and characteristic ij are cut out by the web of quartic monoids

$$aP_{i,k}P_{k,j} + bP_{i,l}P_{l,j} + cP_{i,m}P_{m,j} + dP_{inr}P_{iklm} = 0$$

with the multiplicities $(i^3j^1k^2l^2m^2n^2r^2)$ and the common curves $F_{ik}, F_{il}, F_{im}, F_{in}, F_{ir}, F_{j8}$. The curves of the web have order 16, the multiplicities $(i^3j^3k^4l^4m^4n^4r^4)$, and genus 7, as follows from the equivalence

$$4\pi \equiv C^{16} + 3E_{i8} + E_{j8} + 2(E_{k8} + \cdots + E_{r8}) + F_{ik} + \cdots + F_{ir} + F_{j8}.$$

The curves C^{16} meet the rational curves on just one of their triple points each once, those on both or neither of these not at all.

Theorem (e): The curves determined on Δ by the O-functions of order 2 and characteristic ij are cut out by the net of quintics

$$aP_{l,k}P_{k,lmnr} + bP_{m,l}P_{l,knr} + cP_{m,k}P_{k,lnr} = 0$$

with the multiplicities $(i^2j^2k^2l^2m^3n^3r^3)$ and the common curves $F_{k8}, F_{l8}, E_{kl}, F_{km}, F_{kn}, F_{kr}, F_{lm}, F_{ln}, F_{lr}, F_{mn}, F_{nr}, F_{rm}$. The curves of the net thus have order 11, multiplicities $(i^3j^3k^2l^2m^2n^2r^2)$, and genus 6, as follows from the equivalence $5\pi \equiv K^{11} + 2E_{i8} + 2E_{j8} + \cdots + 3E_{r8} + F_{k8} + \cdots + F_{rm}$.

The curves K^{11} are found to meet the rational curves on just one of their triple points not at all, those on both or neither of these in one point each.

Finally, we find bases for the O and E functions of order 2 and characteristic ijk8 to be, respectively,

(27) $\vartheta_{mi}\vartheta_{mjk8}$; $\vartheta_{mj}\vartheta_{mik8}$; $\vartheta_{mk}\vartheta_{mkj8}$; $\vartheta_{m8}\vartheta_{nrl8}$

(28) $\vartheta_{ij}\vartheta_{k8}$; $\vartheta_{jk}\vartheta_{i8}$; $\vartheta_{ki}\vartheta_{j8}$

from which, by the usual processes, we have

Theorem (f): The curves determined on Δ by the O-functions of order 2 and characteristic ijk8 are cut out by the web of cubic surfaces

$$aP_{i,m}P_{mjk} + bP_{j,m}P_{mik} + cP_{k,m}P_{kij} + dP_{mijk} = 0$$

which have at the points of Q_7^3 the multiplicities $(i^2j^2k^2l^1m^1n^1r^1)$ and the common curves F_{ij}, F_{jk}, F_{ki}. The curves of the web thus have order 15, the

multiplicities $(i^4j^4k^4l^3m^3n^3r^3)$, and genus 7, as follows from

$$3\pi \equiv K^{15} + 2E_{i8} + 2E_{j8} + 2E_{k8} + E_{l8} + \cdots + E_{r8} + F_{ij} + F_{jk} + F_{ki}.$$

The curves K^{15} meet the rational curves on 2 or neither of their 4-fold points each once, those on one or all of them not at all.

Theorem (g): The curves determined on Δ by the E-functions of order 2 and characteristic ijk8 are cut out by the net of quartics

$$aP_{i,j}P_{j,i} + bP_{i,j}P_{j,k} + cP_{i,k}P_{j,i} = 0$$

with the multiplicities $(i^2j^2k^2l^2m^2n^2r^2)$ and the common curves $E_{ij}, F_{il}, F_{im}, F_{in}, F_{ir}, F_{jl}, F_{jm}, F_{jn}, F_{jr}$. The curves of the net have order 12, the multiplicities $(i^2j^2k^2l^3m^3n^3r^3)$, and genus 6, as follows from

$$4\pi \equiv K^{12} + 2E_{i8} + 2E_{j8} + E_{k8} + 2E_{l8} + \cdots + 2E_{r8} + E_{ij} + F_{il} + \cdots + F_{jn}.$$

The curves K^{12} meet the rational curves on 2 or none of their double points not at all, those on one or all of their double points each once.

From the equivalences of this section, we see that the curves determined on Δ by the ϑ-functions of order 2 and given characteristic are all dependent on the 29 curves π, F_{ij}, F_{i8}. The procedure by which this result was obtained makes it clear that the curves determined on Δ by the ϑ-functions of given order and given rational characteristic with denominator 2, depend on π, F_{ij}, and F_{i8}. (This result does not necessarily apply to the separate parts of any composite curves that may appear.)

A great many additional interesting results have been obtained by subjecting the above linear systems to further conditions.

Let now the triple points q_i, q_j, q_k, q_l of Δ be taken as vertices A_1, A_2, A_3, A_4 of the reference tetrahedron. Then the cubic involution, $\rho y_i' = y_i^{-1}$ $(i = 1,2,3,4)$, is denoted by A_{ijkl}. The transformation A_{ijkl} takes the dianodal Δ into the dianodal Δ' of the $Q_7'^3$ which is the transform of Q_7^3. Applying these involutions to the above results, we find that the three webs of curves C^{12}, K^{15}, C^{16} are not essentially distinct from the Cremona point of view. I.e., a web of one kind on Δ may be transformed into a web of any other kind on Δ'. Similarly for the nets C^{15}, K^{12}, K^{11}.

These Cremona transformations A_{ijkl}, and the corresponding ϑ-involutions I_{ijkl} (Coble, op. cit., Ch. IV.) yield new proofs of many of the above results, and reveal significant relationships between the algebraic and transcendental methods of treating the curves on Δ.

VITA

Franz Edward Hohn was born September 4, 1915, in Warrenton, Missouri, where he attended the public and high schools. He received the first two years of his college education at Central Wesleyan Junior College, Warrenton, Missouri. His undergraduate work was completed at McKendree College, Lebanon, Illinois, where he graduated in June, 1936, with the B.S. degree.

In June, 1937, he received his M.S. degree at the University of Illinois. He was a scholar there during 1936-1937, a fellow during 1937-1938, and a teaching assistant during 1938-1939 and 1939-1940. He has been engaged as instructor for the year 1940-1941 by the University of Arizona.

He is a member of Pi Mu Epsilon, the Society of Sigma Xi, and the American Mathematical Society.